Theories with global gauge anomalies on the lattice

P. Mitra[a] *

[a]Saha Institute of Nuclear Physics, Block AF, Bidhannagar, Calcutta 700 064, INDIA

A global anomaly in a chiral gauge theory manifests itself in different ways in the continuum and on the lattice. In the continuum case, functional integration of the fermion determinant over the whole space of gauge fields yields zero. In the case of the lattice, it is not even possible to define a fermion measure over the whole space of gauge configurations. However, this is not necessary, and as in the continuum, a reduced functional integral is sufficient for the existence of the theory.

Presented at Lattice 2000, Bangalore hep-lat/0010003

1. Introduction

Anomalies are of two different types. Local or divergence anomalies have been known since 1969 [1]: classically conserved symmetry currents cease to be conserved after quantization if there are anomalies of this kind. For example, the theory

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - eA\!\!\!/)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

does not obey the expected conservation law for its axial current:

$$\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) = \frac{e^2}{16\pi^2}F_{\mu\nu}F^{\mu\nu} \neq 0. \tag{2}$$

If an anomalous current is associated with a *gauged symmetry*, it leads to an apparent problem in quantization because the equations of motion of the gauge fields require the current to be conserved. A treatment of such a theory just like a usual gauge theory shows an inconsistency. This problem can be sorted out by paying proper attention to phase space constraints, as suggested by [2]. The anomaly itself can be made to vanish in a sense by going to the constrained submanifold of classical phase space. However, theories with anomalous gauge currents are to be distinguished from theories with nonanomalous gauge currents. If a theory is *nonanomalous*, it possesses gauge freedom, and is describable in any one of an infinite variety of gauges, whereas an *anomalous gauge theory* may have its gauge fixed by the anomaly itself.

The second kind of anomaly is the *topological or global* anomaly discovered in 1982 [3]. The gauge current is conserved here, but the topology of the gauge group in the continuum is such that the fermion determinant changes under *large* gauge transformations *i.e.,* those not continuously connected to the identity transformation. This may result in a *vanishing of the partition function.* An example is provided by the SU(2) gauge theory with a doublet of Weyl fermions. Here gauge transformations fall into two disconnected classes, those connected to the identity, and the others. The fermion determinant picks up a change of sign if the gauge field in which it is evaluated is transported along a (non-gauge) path in the gauge configuration space connecting two large-gauge-transformation-related configurations. Such theories nevertheless exist in the continuum, as shown in [4] by appealing to canonical quantization.

After the recent developments around the Ginsparg-Wilson relation for lattice fermions, a general formulation of global anomalies on the lattice has been given [5]. Our aim here is to show that the theories exist on the lattice as well, though an explicit construction cannot be given at the moment.

The continuum argument of [4] is recapitulated in the next section. Then we pass to the lattice.

*e-mail mitra@tnp.saha.ernet.in

2. Functional integral in the continuum

The full partition function of the gauge theory with fermions may be written as

$$Z = \int \mathcal{D}A Z[A], \tag{3}$$

$$Z[A] \equiv e^{-S_{eff}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S(\psi,\bar{\psi},A)} \tag{4}$$

An anomaly-free theory has $Z[A]$ gauge invariant. If there is a gauge anomaly, $Z[A]$ varies under gauge transformations of A:

$$Z[A^g] = e^{i\alpha(A,g^{-1})} Z[A], \tag{5}$$

where α may be regarded as an integral representation of the anomaly. It obeys some consistency conditions (mod 2π):

$$\begin{aligned} \alpha(A, g_2^{-1}g_1^{-1}) &= \alpha(A, g_1^{-1}) + \alpha(A^{g_1}, g_2^{-1}) \\ \alpha(A, g^{-1}) &= -\alpha(A^g, g). \end{aligned} \tag{6}$$

The case becomes one of a global anomaly if α is independent of A, and vanishes for g connected to the identity but not for some g which cannot be continuously connected to the identity. A-independence implies an abelian representation satisfying

$$\alpha(g_2 g_1) = \alpha(g_1) + \alpha(g_2). \tag{7}$$

In the SU(2) case, the two components of the gauge group manifest themselves in two possible values of the phase: $e^{i\alpha} = \pm 1$.

In an anomaly-free theory, the partition function factorizes into the volume of the gauge group and the gauge-fixed partition function:

$$\begin{aligned} Z &= \int \mathcal{D}A Z[A] \\ &= \int \mathcal{D}A Z[A] \int \mathcal{D}g \delta(f(A^g))\Delta_f(A) \\ &= \int \mathcal{D}g \int \mathcal{D}A Z[A^{g^{-1}}]\delta(f(A))\Delta_f(A) \\ &= \int \mathcal{D}g \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A) \\ &= (\int \mathcal{D}g) Z_f \end{aligned} \tag{8}$$

This is the standard Faddeev-Popov argument. Here, $\delta(f)$ represents a gauge-fixing operation and Δ_f is the corresponding Faddeev-Popov determinant.

This decoupling of gauge degrees of freedom does not occur if a local anomaly is present. For a global anomaly however, the partition function factorizes again:

$$Z = \int \mathcal{D}g e^{-i\alpha(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A) \tag{9}$$

As the phase factors form a representation of the gauge group,

$$\begin{aligned} \int \mathcal{D}g e^{-i\alpha(g)} &= \int \mathcal{D}(gh) e^{-i\alpha(gh)} \\ &= e^{-i\alpha(h)} \int \mathcal{D}g e^{-i\alpha(g)} \end{aligned} \tag{10}$$

where h stands for a fixed gauge transformation. If h is *not* connected to the identity, $e^{-i\alpha(h)} \neq 1$, and consequently $\int \mathcal{D}g e^{-i\alpha(g)} = 0$, which in turn means that $Z = 0$. Does this mean that the theory cannot be defined? Let us look at expectation values of gauge invariant operators.

$$\begin{aligned} &\frac{\int \mathcal{D}A Z[A]\mathcal{O}}{\int \mathcal{D}A Z[A]} = \\ &\frac{\int \mathcal{D}g e^{-i\alpha(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)\mathcal{O}}{\int \mathcal{D}g e^{-i\alpha(g)} \int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)} \end{aligned} \tag{11}$$

The expression on the right is of the form $\frac{0}{0}$ because the factor $\int \mathcal{D}g e^{-i\alpha(g)}$ vanishes, as we have seen above. Can one not reinterpret the ratio by removing this common vanishing factor?

$$< \mathcal{O} > \stackrel{?}{=} \frac{\int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)\mathcal{O}}{\int \mathcal{D}A Z[A]\delta(f(A))\Delta_f(A)}. \tag{12}$$

The right hand side is precisely what one gets in the *canonical* approach to quantization where gauge degrees of freedom are removed by fixing the gauge at the classical level and only physical degrees of freedom enter the functional integral. The Faddeev-Popov determinant arises in the canonical approach as the determinant of the matrix of Poisson brackets of what may be called the "second class constraints", *i.e.*, the Gauss law operator and the gauge fixing condition f, which is of course introduced by hand and not really a constraint of the theory. There are both ordinary fields and conjugate momenta, but the latter are easily integrated over. The point is that the full functional integral is not needed in the canonical approach and there is no harm if it vanishes!

A trace is left behind by the global anomaly. One may imagine a classification of the gauge-fixing functions f where f, f' are said to belong to the same class if there exists a gauge transformation connected to the identity to go from a configuration with $f = 0$ to one with $f' = 0$. Then $Z_f = Z_{f'}$. More generally, when such a transformation is not connected to the identity,

$$Z_f = e^{-i\alpha(g_0)} Z_{f'}, \tag{13}$$

where g_0 is determined by f, f'. These factors $e^{-i\alpha(g_0)}$ occurring in partition functions cancel out in expectation values of gauge invariant operators, so that Green functions of gauge invariant operators are fully gauge independent [4].

There is an assumption in all this: that there is a possibility of fixing the gauge. A general theorem [6] asserts that gauges *cannot* be fixed in a smooth way. For the construction of functional integrals, however, it is sufficient to have *piecewise smooth gauges*. It should also be remembered that these questions arise even for theories *without* disconnected gauge groups and are not specific to the context of global anomalies.

3. Lattice formulation

On going to the lattice, one starts to use group-valued variables associated with links instead of A defined at points of the continuum. The topology also changes: the gauge group becomes *connected* on the lattice: it becomes possible to go to any gauge transformation from the identity in a continuous manner. Thus there are no large gauge transformations any more. Does it mean that there is no global anomaly on the lattice? The issue is complicated because chiral symmetry is not straightforward here. Chiral symmetry on the lattice has begun to make more sense in the last few years thanks to the Ginsparg-Wilson relation imposed on D, the euclidean lattice Dirac operator:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D, \tag{14}$$

where a is the lattice spacing. An analogue of γ_5 appears from the above relation:

$$\gamma_5 D = -D\Gamma_5,\ \Gamma_5 \equiv \gamma_5(1 - aD). \tag{15}$$

It satisfies

$$(\Gamma_5)^2 = 1,\ (\Gamma_5)^\dagger = \Gamma_5, \tag{16}$$

and can be used to define left-handed projection:

$$\begin{aligned} P_-\psi &\equiv \frac{1}{2}[1 - \Gamma_5]\psi = \psi \\ \bar\psi P_+ &\equiv \bar\psi\frac{1}{2}[1 + \gamma_5] = \bar\psi. \end{aligned} \tag{17}$$

In this way of defining chiral projections, P_-, but not P_+, depends on the gauge field configuration. Nontriviality of chirality on the lattice stems from this P_-.

A fermion measure is defined by specifying a basis of lattice Dirac fields $v_j(x)$ satisfying

$$P_- v_j = v_j,\ (v_j, v_k) = \delta_{jk}. \tag{18}$$

One has to integrate over Grassmann-valued expansion coefficients in

$$\psi(x) = \sum_j a_j v_j(x). \tag{19}$$

Expansion coefficients also come from the expansion of $\bar\psi$ in terms of $\bar v_j$ satisfying $\bar v_j P_+ = \bar v_j$, but these are as usual, *i.e.*, do not involve gauge fields.

Questions of *locality* and *integrability* arise because of the gauge field dependence in P_-. Absence of a local anomaly appears to be sufficient to ensure locality [7]. Global anomalies are manifested as a lack of *integrability*.

Consider, following [5], a closed path in the SU(2) gauge configuration space, with the parameter t running from 0 to 1. Define

$$f(t) = \det[1 - P_+ + P_+ D(t) Q_t D(0)^\dagger], \tag{20}$$

with $D(t)$ the Dirac operator corresponding to gauge fields at parameter value t, and Q_t the unitary transport operator for $P_-(t)$ defined by

$$\partial_t Q_t = [\partial_t P_-(t), P_-(t)]Q_t,\ Q_0 = 1. \tag{21}$$

Then $f(t)$ is real, positive and satisfies

$$f(1) = \mathcal{T} f(0). \tag{22}$$

Here

$$\mathcal{T} = \det[1 - P_-(0) + P_-(0)Q_1] = \pm 1 \tag{23}$$

depending on the topology of the considered path in the gauge configuration space. f changes sign an even or odd number of times along path depending on $\mathcal{T}$ and while $\det D(t)$ is related to f^2,

$$\det D(t) \det D(0)^\dagger = f^2(t), \tag{24}$$

the chiral fermion determinant $\det D_\chi(t)$ behaves like f:

$$\det D_\chi(t) \det D_\chi(0)^\dagger = f(t) W(t)^{-1},$$
$$(D_\chi)_{ij} \equiv a^4 \sum_x \bar{v}_i(x) D v_j(x). \tag{25}$$

Here $W(t)$ is a phase factor arising from the gauge field dependence of v_j. It is a lattice artifact and may be taken to reduce to unity near the continuum limit.

Then $\det D_\chi$ changes sign, *i.e.*, fails to return to its starting value after transportation along a closed path if the path has

$$\mathcal{T} = -1. \tag{26}$$

Such paths have been shown to exist in the SU(2) theory. A part of such a path lies along a gauge orbit, and a part is non-gauge.[2]

Thus $\det D_\chi$ is multivalued, implying that the fermion measure is not well defined, and hence the functional integral does not make sense. This is roughly similar to the continuum. The Dirac operator is gauge-invariant and its determinant and f can change only on non-gauge portions of the closed path. So the problem of sign change of f occurs once again in non-gauge paths connecting gauge-related configurations. However, in the continuum, the sign change occurs between configurations which can be connected only by a non-gauge path. On the lattice, the sign change occurs when configurations are connected by a non-gauge path, though a connection is also possible by a gauge path, with no accompanying change of sign. In other words, the sign or phase is ambiguous. There is an obvious remedy: to restrict the functional integral to one point on each orbit, *i.e.*, to fix the gauge. Problematic paths with $\mathcal{T} = -1$ can thereby be avoided. The fermion measure is defined over the reduced configuration space and the theory exists.

Note that W need not be taken to be $1 + \mathcal{O}(a)$. This may yield an alternative way of defining the theory on a finite lattice.

As in the continuum, there is the question whether the gauge can be fixed. In the lattice literature, there is a lot of discussion on gauge fixing, Gribov ambiguities and related phenomena. Without going into these details, one can see that if one considers a single gauge orbit as a set, it will in principle be possible to pick a configuration from that set, and this fixes the gauge for a single orbit. The procedure can be repeated for each orbit. A problem arises only when one tries to combine these choices for different orbits. A smooth gauge is not expected to exist, but a piecewise smooth choice should be possible, as in the continuum. There are hopes of making gauge choices which lead to BRST invariance in the continuum limit [8].

[2]The integrability condition valid for such paths is $W(1) = \mathcal{T}$, but in the near-continuum region considered in the current literature, $W = 1 + \mathcal{O}(a)$ cannot become -1.

Acknowledgment

I would like to thank Maarten Golterman for some comments on gauge fixing on the lattice.

REFERENCES

1. S. Adler, Phys. Rev. 177 (1969) 2426; J. S. Bell and R. Jackiw, Nuov. Cim. 60A (1969) 47.
2. L. D. Faddeev, Phys. Lett. 145B (1984) 81.
3. E. Witten, Phys. Lett. 117B (1982) 324.
4. P. Mitra, Lett. Math. Phys. 31 (1994) 111; see also P. Mitra, hep-th/9506158.
5. O. Bär and I. Campos, DESY-99/188.
6. I. M. Singer, Comm. Math. Phys. 60 (1978) 7.
7. M. Lüscher, hep-lat/9904009.
8. M. Testa, hep-lat/9803025.